

A formulation of gravity as non-linear electrodynamics

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A very general description of non-linear classical gravity, using generalized constitutive equations and constitutive tensors, is presented. Our approach includes non-Lagrangian as well as Lagrangian theories, allows for gravitational fields in the widest possible variety of media and accommodates the incorporation of non-local effects. Eventually, equations for non-linear gravito-electromagnetic waves are studied and solved in detail.

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I. INTRODUCTION

Many efforts in the literature, over several decades, have been devoted to the construction of gauge theories of gravity, mainly motivated [1] by:

- (i) conceptual simplicity within the geometric framework of fibre-bundle picture of modern field theories;
- (ii) the search for quantum gravity (see [2] and the references therein), that should unify quantum theory and general relativity. Such an unification is logically compelling if one thinks that gravity couples to the energy-momentum tensor of matter, but matter fields are defined and treated via quantum theory, and the semiclassical analysis where gravity remains classical is a hybrid scheme that cannot hold at all energy scales.
- (iii) attempts of building unified theories of all fundamental interactions. The various models can be classified according to the relevant structure group in the fibre [1]. Within the Poincaré group, this has been either the translations, or the Lorentz group, or the whole Poincaré group. The enlargement of the group has involved using the conformal, or special affine or affine groups. Moreover, the so-called supergroups have included the graded-Poincaré or the superconformal group. Interestingly, re-expressing the action of supergravity in Yang–Mills form has made it possible to simplify enormously the proof of its invariance under local supersymmetry transformations [3].

An alternative view is also available, which is convenient, if not natural, if one is interested in the geometry of the space of histories and its application to functional integrals in quantum field theory [4]. Within this framework, gauge theories are all field theories whose action functional S is invariant under an infinite-dimensional Lie group. Gauge invariance is then expressed by the existence of vector fields Q_α (α being a Lie-algebra index) on the space of histories that leave invariant S , i.e.

$$Q_\alpha S = 0. \quad (1.1)$$

All known gauge theories belong to one of the following families:

- (i) Type I: The Lie bracket $[,]$ of such vector fields is a linear combination of the Q_α themselves, i.e.

$$[Q_\alpha, Q_\beta] = C^\gamma_{\alpha\beta} Q_\gamma, \quad C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}, \quad (1.2)$$

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the coefficients $C_{\alpha\beta}^\gamma$ being structure constants, hence independent of the field variables φ^i .

Interestingly, Maxwell's electrodynamics, Yang–Mills theory and Einstein's gravity share the property of being type-I theories, which was overlooked for a long time and even nowadays, since Yang–Mills theory was first formulated in Minkowski space-time, and only there can its perturbative renormalizability be proved, this property being spoiled by space-time curvature [4].

Type II: The structure constants of the previous equation are replaced by structure functions $\tilde{C}_{\alpha\beta}^\gamma$, i.e. dependent on field variables, so that

$$\frac{\delta}{\delta\varphi^i}\tilde{C}_{\alpha\beta}^\gamma \neq 0. \quad (1.3)$$

An example is given by simple supergravity in 4 space-time dimensions, endowed with auxiliary fields.

Type III: The right-hand side of the defining equation acquires also an inhomogeneous term, so that

$$[Q_\alpha, Q_\beta] = \tilde{C}_{\alpha\beta}^\gamma Q_\gamma + T_{\alpha\beta}^i \frac{\delta S}{\delta\varphi^i}, \quad (1.4)$$

where T is skew-symmetric in the Lie-algebra indices. An example is provided by simple and extended supergravity in any number of space-time dimensions, without auxiliary fields.

Now we can revert to a topic more immediately related to the calculations we are interested in, i.e. the formulation of Einstein's theory as $SL(2, C)$ gauge theory, following [5]. We study in detail the equations for nonlinear gravito-electromagnetic waves and their solutions.

II. ELECTRODYNAMICS AND GENERAL NONLINEAR CONSTITUTIVE EQUATIONS

To set the context and specify notation, we review established results in this section [6]. We consider only flat spacetime, so that the metric is given by the Minkowski tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$, $x^\mu = (ct, x^i)$, $\mu, \nu, \dots = 0, 1, 2, 3$; $i, j, \dots = 1, 2, 3$; with $\partial_\mu = \partial/\partial x^\mu = [c^{-1}\partial/\partial t, \nabla]$. The antisymmetric Levi-Civita tensor is written $\varepsilon^{\mu\nu\rho\sigma}$, with $\varepsilon^{0123} = 1$. With tensor notation, we denote by $F_{\mu\nu}$ the electromagnetic field tensor and by $G_{\mu\nu}$ the tensor expressing the constitutive equations (see below), while their Hodge dual tensors are $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$, and hence Maxwell's equations become

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu G^{\mu\nu} = j^\nu, \quad (2.1)$$

where $j^\mu = (c\rho, \mathbf{j})$ is the 4-current. The first equations in (2.1) imply $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is an Abelian gauge field; but in general there is no similar representation for $G_{\mu\nu}$. With covariant notation we have 2 independent invariants

$$X = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad Y = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}. \quad (2.2)$$

Then the nonlinear constitutive equation of [7, 8] is

$$G_{\mu\nu} = N(X, Y)F_{\mu\nu} + cM(X, Y)\tilde{F}_{\mu\nu}. \quad (2.3)$$

Although the constitutive equations (2.3) are fairly general, they do not take account of a variety of possibilities, such as anisotropic media [9], chiral materials where derivative terms enter [10], piezoelectric and ferromagnetic materials, and so forth. Therefore it was proposed [6] to generalize (2.3) by introducing the *constitutive tensors* $S_{\mu\nu}$, $R_{\mu\nu}^{\rho\sigma}$, and $Q_{\mu\nu}^{\rho\sigma\lambda_1\dots\lambda_n}$, $n = 1, 2, 3, \dots$ and write a general nonlinear constitutive equation

$$G_{\mu\nu} = S_{\mu\nu} + R_{\mu\nu}^{\rho\sigma}F_{\rho\sigma} + Q_{\mu\nu}^{\rho\sigma\lambda_1}D_{\lambda_1}F_{\rho\sigma} \quad (2.4)$$

$$+ Q_{\mu\nu}^{\rho\sigma\lambda_1\lambda_2}D_{\lambda_1}D_{\lambda_2}F_{\rho\sigma} + \dots + Q_{\mu\nu}^{\rho\sigma\lambda_1\dots\lambda_n}D_{\lambda_1}D_{\lambda_2}\dots D_{\lambda_n}F_{\rho\sigma}. \quad (2.5)$$

Evidently the formula (2.4), taken together with Maxwell's equations, includes all possibilities discussed so far, as well as new ones, defining the general nonlinear electromagnetic theory. We impose Lorentz covariance on the constitutive equations, the constitutive tensors will depend on the fields through the invariants X and Y , as follows: $S_{\mu\nu}$ is a constant independent of X and Y , while

$$R_{\mu\nu}^{\rho\sigma} = R_{\mu\nu}^{\rho\sigma}(X, Y), \quad Q_{\mu\nu}^{\rho\sigma\lambda_1\dots\lambda_n} = Q_{\mu\nu}^{\rho\sigma\lambda_1\dots\lambda_n}(X, Y, \dots), \quad (2.6)$$

where “...” denotes covariant derivatives of the invariants X, Y up to n -th order. Obviously $S_{\mu\nu}$ is antisymmetric, $R_{\mu\nu}^{\rho\sigma}$ is antisymmetric in its upper and lower indices separately, and the $Q_{\mu\nu}^{\rho\sigma\lambda_1\ldots\lambda_n}$ are antisymmetric in their upper and first two lower indices; with respect to the λ_i , they are symmetric Lorentz tensors.

If the equations of motion for the nonlinear theory derive from a Lagrangian $L(X, Y)$ which is a scalar function of the invariants X and Y but does not depend on their derivatives, then from the usual definitions together with Eq. (2.3) we have

$$G_{\mu\nu} = -\frac{\partial L(X, Y)}{\partial X} F_{\mu\nu} - \frac{\partial L(X, Y)}{\partial Y} \tilde{F}_{\mu\nu}. \quad (2.7)$$

Comparing (2.4) and (2.7) gives us the constitutive tensors,

$$\begin{aligned} S_{\mu\nu} &= 0, \\ R_{\mu\nu}^{\rho\sigma} &= -\frac{\partial L(X, Y)}{\partial X} \delta_{[\mu}^{\rho} \delta_{\nu]}^{\sigma} - \frac{\partial L(X, Y)}{\partial Y} \varepsilon_{[\mu\nu]\lambda\delta} \eta^{\lambda\rho} \eta^{\delta\sigma}, \\ Q_{\mu\nu}^{\rho\sigma\lambda_1\ldots\lambda_n} &= 0. \end{aligned} \quad (2.8)$$

Now we can relate this formulation of nonlinear electrodynamics with gravity.

III. GRAVITY AS A GAUGE THEORY

Here we consider the formulation of the Einstein gravity as $SL(2, C)$ gauge theory [5]. The gauge potentials are interpreted in terms of an affine connection on a complex vector bundle. In the $SL(2, C)$ gauge theory they are taken to be the dyad components of the spinor connection, and are therefore the Newman–Penrose spin coefficients.

Let $\Gamma_{\mu}^A{}_B$ be the complex components of the spinor connection ∇ , i.e.

$$\nabla_{\mu} \xi_a^A = \Gamma_{\mu}^A{}_B \xi_a^B, \quad (3.1)$$

where ξ_a^A is a spinor basis of the complex vector bundle. The dyad components of the spinor connection are defined by

$$B_{\mu a}{}^b \equiv \Gamma_{\mu A}{}^B \xi_a^A \xi^b_B. \quad (3.2)$$

The field strength of the gauge theory (spinor curvature tensor) can be expressed through Γ_{μ}^{AB} as

$$\hat{F}_{\mu\nu A}{}^B = \partial_{\nu} \Gamma_{\mu A}{}^B - \partial_{\mu} \Gamma_{\nu A}{}^B + \Gamma_{\mu A}{}^C \Gamma_{\nu C}{}^B - \Gamma_{\nu A}{}^C \Gamma_{\mu C}{}^B. \quad (3.3)$$

Thus, we can rewrite $\hat{F}_{\mu\nu A}{}^B$ directly through the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ as follows:

$$\hat{F}_{\mu\nu a}{}^b = \frac{1}{2} R^{\alpha}{}_{\beta\mu\nu} \sigma_{\alpha a c'} \sigma^{\beta b c'}, \quad (3.4)$$

where $\sigma_{\alpha ab'} \equiv \sigma_{\alpha AB'} \xi_a^A \bar{\xi}_{b'}^{B'}$, the $\sigma_{\alpha AB'}$ being the Infeld–van der Waerden symbols which turn tensors into spinors and express the isomorphism between the tangent space at a point of space-time and the tensor product of unprimed spin-space and primed spin-space. In Minkowski space-time they reduce to the Pauli spin matrices.

The field strength obeys the Bianchi identity

$$\nabla_{\rho} \hat{F}_{\mu\nu A}{}^B + \text{permutation} = 0, \quad (3.5)$$

which are indeed the second set of the Maxwell equations. The Riemann tensor $R_{\alpha\beta\mu\nu}$ can be decomposed into the Weyl tensor $W_{\alpha\beta\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + g_{\mu[\rho} R_{\sigma]\nu} + g_{\nu[\rho} R_{\sigma]\mu} + \frac{1}{2} R g_{\mu[\rho} g_{\sigma]\nu}. \quad (3.6)$$

The Einstein equations are

$$R_{\mu\nu} = k \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \quad (3.7)$$

Thus, the decomposition (3.6) becomes

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + 2kg_{[\rho[\mu}R_{\nu]\sigma]} - \frac{1}{2}kg_{\mu[\sigma}g_{\rho]\nu}T. \quad (3.8)$$

A corresponding expression can be written for the field-strength tensor

$$\hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}^W + k\tau_{\mu\nu}, \quad (3.9)$$

$$F_{\mu\nu} = G_{\mu\nu} - S_{\mu\nu} \quad (3.10)$$

$$G_{\mu\nu} = S_{\mu\nu} + R_{\mu\nu}^{\rho\sigma}F_{\rho\sigma} + 0 \implies \quad (3.11)$$

$$\text{Einstein gravity } R_{\mu\nu}^{\rho\sigma} = \delta_{\mu}^{\rho}\delta_{\nu}^{\sigma}, S_{\mu\nu} = -k\tau_{\mu\nu}, Q = 0. \quad (3.12)$$

This suggests the general structure

$$S_{\mu\nu} = C(x)\delta_{[\mu}^{\rho}\delta_{\nu]}^{\sigma}\varepsilon_{[\mu\nu]\lambda\delta}, \quad (3.13)$$

$$R_{\mu\nu}^{\rho\sigma} = A(X, Y)\delta_{[\mu}^{\rho}\delta_{\nu]}^{\sigma} + B(X, Y)\varepsilon_{[\mu\nu]\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}, \quad (3.14)$$

while Q is an as yet unspecified tensor $Q_{\mu\nu}^{\rho\sigma\lambda}$.

Let us introduce the covariant derivative

$$D_{\mu}^{(B)}X = \partial_{\mu}X - [B_{\mu}, X], \quad (3.15)$$

then we can write the Bianchi identity as

$$\varepsilon^{\mu\nu\rho\sigma}D_{\nu}^{(B)}\hat{F}_{\rho\sigma} = 0. \quad (3.16)$$

By using the formula (3.9) we obtain

$$\varepsilon^{\mu\nu\rho\sigma}D_{\nu}^{(B)}\hat{F}_{\rho\sigma}^W = kJ^{\mu}, \quad (3.17)$$

where J^{μ} is the current corresponding to the matter

$$J^{\mu} = -\varepsilon^{\mu\nu\rho\sigma}D_{\nu}^{(B)}\tau_{\rho\sigma} \sim T. \quad (3.18)$$

Now we are ready to compare the above formulas with the standard nonlinear electrodynamics: the Maxwell equations (2.1) and the constitutive equations (2.4). We can identify (2.4) with (3.9) and obtain

$$G_{\mu\nu} = \hat{F}_{\mu\nu}^W, \quad (3.19)$$

$$F_{\mu\nu} = \hat{F}_{\mu\nu}, \quad (3.20)$$

$$S_{\mu\nu} = -k\tau_{\mu\nu}, \quad (3.21)$$

$$j^{\mu} = kJ^{\mu}. \quad (3.22)$$

It follows from (3.17) that

$$\varepsilon^{\mu\nu\rho\sigma}D_{\nu}^{(B)}G_{\rho\sigma} = j^{\mu}. \quad (3.23)$$

To obtain the analogy with the first set of the standard Maxwell equations $\partial_{\mu}G^{\mu\nu} = j^{\nu}$, we introduce the Hodge dual tensor

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}. \quad (3.24)$$

Then we have

$$D_{\nu}^{(B)}\tilde{G}^{\mu\nu} = \frac{1}{2}j^{\mu}. \quad (3.25)$$

IV. NONLINEAR GRAVITO-ELECTROMAGNETISM

Here we will turn to 3-dimensional notation for convenience and examples. Let us consider the gravito-electromagnetism in the weak-field approximation (following, e.g., [11]). Recall the standard Maxwell equations in SI units [12]

$$\begin{aligned}\text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \text{div } \mathbf{B} &= 0, \\ \text{curl } \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, & \text{div } \mathbf{D} &= \rho,\end{aligned}\tag{4.1}$$

where \mathbf{E} is the electric field, \mathbf{B} is the magnetic field, ρ is charge density, \mathbf{j} is electric current density. In the linear case

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E},\tag{4.2}$$

In the nonlinear case these equations can be presented in the form [13]

$$\begin{aligned}\mathbf{D} &= M(I_1, I_2) \mathbf{B} + \frac{1}{c^2} N(I_1, I_2) \mathbf{E}, \\ \mathbf{H} &= N(I_1, I_2) \mathbf{B} - M(I_1, I_2) \mathbf{E},\end{aligned}\tag{4.3}$$

where the invariants are

$$I_1 = \mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2, \quad I_2 = \mathbf{B} \cdot \mathbf{E}.\tag{4.4}$$

Their gravitational analogues in SI are

$$\text{curl } \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div } \mathbf{B}_g = 0,\tag{4.5}$$

$$\text{curl } \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t} + \frac{1}{\varepsilon_g c^2} \mathbf{j}_g, \quad \text{div } \mathbf{E}_g = \frac{1}{\varepsilon_g} \rho_g,\tag{4.6}$$

where \mathbf{E}_g is the static gravitational field (conventional gravity, also called gravitoelectric for the sake of analogy), \mathbf{B}_g is the gravitomagnetic field, ρ_g is mass density, \mathbf{j}_g is mass current density, G is the gravitational constant, ε_g is the gravity permittivity (analog of ε_0). Here

$$\varepsilon_g = -\frac{1}{4\pi G}, \quad \mu_g = -\frac{4\pi G}{c^2},\tag{4.7}$$

are the gravitational permittivity and permeability, respectively.

The main idea is to introduce analogues of \mathbf{H} and \mathbf{D} to write (4.6) in the Maxwell form for 4 fields in SI as

$$\text{curl } \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div } \mathbf{B}_g = 0,\tag{4.8}$$

$$\text{curl } \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t} + \mathbf{j}_g, \quad \text{div } \mathbf{D}_g = \rho_g.\tag{4.9}$$

In the linear-gravity case

$$\mathbf{D}_g = \varepsilon_g \mathbf{E}_g,\tag{4.10}$$

$$\mathbf{B}_g = \mu_g \mathbf{H}_g,\tag{4.11}$$

$$\varepsilon_g \mu_g = \frac{1}{c^2}.\tag{4.12}$$

The main idea is the following: the linear-gravity case (4.10)–(4.12) corresponds to weak approximation and some special case of gravitational field configuration. We generalize it to nonlinear case which can explain other configurations and non-weak fields, as in (4.3), by

$$\mathbf{D}_g = M_g(I_{g1}, I_{g2}) \mathbf{B}_g + \frac{1}{c^2} N_g(I_{g1}, I_{g2}) \mathbf{E}_g,\tag{4.13}$$

$$\mathbf{H}_g = N_g(I_{g1}, I_{g2}) \mathbf{B}_g - M_g(I_{g1}, I_{g2}) \mathbf{E}_g,\tag{4.14}$$

where the invariants are

$$I_{g1} = \mathbf{B}_g^2 - \frac{1}{c^2} \mathbf{E}_g^2, \quad I_{g2} = \mathbf{B}_g \cdot \mathbf{E}_g.\tag{4.15}$$

The gravity-Maxwell equations (4.8)–(4.9) together with the nonlinear gravity-constitutive equations (4.13)–(4.14) can give a nonlinear electrodynamics formulation of gravity (at least its particular cases).

V. LINEAR GRAVITO-ELECTROMAGNETIC WAVES

The gravity-Maxwell equations for gravito-electromagnetic waves (far from sources) are

$$\text{curl } \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \text{div } \mathbf{B}_g = 0, \quad (5.1)$$

$$\text{curl } \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \text{div } \mathbf{D}_g = 0 \quad (5.2)$$

with generic values of permittivity and permeability (4.7). Then

$$\text{curl } \mathbf{E}_g = -\mu_g \frac{\partial \mathbf{H}_g}{\partial t}, \text{div } \mathbf{H}_g = 0, \quad (5.3)$$

$$\text{curl } \mathbf{H}_g = \varepsilon_g \frac{\partial \mathbf{E}_g}{\partial t}, \text{div } \mathbf{E}_g = 0. \quad (5.4)$$

We differentiate the first equation with respect to time: $\text{curl } \frac{\partial}{\partial t} \mathbf{E}_g = -\mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2} \Rightarrow \frac{1}{\varepsilon_g} \text{curl}(\text{curl } \mathbf{H}_g) = -\mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2}$. Since $\text{curl}(\text{curl } \mathbf{H}_g) = \text{grad}(\text{div } \mathbf{H}_g) - \Delta \mathbf{H}_g = -\Delta \mathbf{H}_g$, then

$$\Delta \mathbf{H}_g = \varepsilon_g \mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \quad (5.5)$$

By analogy, from the second equation $\text{curl } \frac{\partial}{\partial t} \mathbf{H}_g = \varepsilon_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2} \Rightarrow \frac{-1}{\mu_g} \text{curl}(\text{curl } \mathbf{E}_g) = \varepsilon_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2}$. Hence we get the wave equation for \mathbf{E}_g ,

$$\Delta \mathbf{E}_g = \varepsilon_g \mu_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2}. \quad (5.6)$$

VI. NONLINEAR GRAVITO-ELECTROMAGNETIC WAVES

The differences begin with the constitutive equations (4.13)–(4.14). For simplicity put $M_g = 0$. Then

$$\mathbf{D}_g = \frac{N}{c^2} \mathbf{E}_g, \quad (6.1)$$

$$\mathbf{B}_g = \frac{1}{N} \mathbf{H}_g \quad (6.2)$$

where $N \equiv N_g(I_{g1}, I_{g2})$. The Maxwell equations become (hereafter the dots denote time derivatives)

$$\text{curl } \mathbf{E}_g = -\left(\frac{1}{N}\right)^{\bullet} \mathbf{H}_g - \frac{1}{N} \frac{\partial \mathbf{H}_g}{\partial t}, \quad (6.3)$$

$$\text{div} \left(\frac{1}{N} \mathbf{H}_g \right) = \mathbf{H}_g \text{grad} \left(\frac{1}{N} \right) + \frac{1}{N} \text{div}(\mathbf{H}_g) = 0, \quad (6.4)$$

$$\text{curl } \mathbf{H}_g = \frac{\dot{N}}{c^2} \mathbf{E}_g + \frac{N}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}, \quad (6.5)$$

$$\text{div} \left(\frac{N}{c^2} \mathbf{E}_g \right) = \mathbf{E}_g \text{grad} \left(\frac{N}{c^2} \right) + \frac{N}{c^2} \text{div}(\mathbf{E}_g) = 0. \quad (6.6)$$

Take derivative of (6.3) with respect to time and get

$$\text{curl } \frac{\partial}{\partial t} \mathbf{E}_g = -\left(\frac{1}{N}\right)^{\bullet\bullet} \mathbf{H}_g - 2\left(\frac{1}{N}\right)^{\bullet} \frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N} \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \quad (6.7)$$

From (6.5) it follows $\frac{\partial \mathbf{E}_g}{\partial t} = \frac{c^2}{N} \text{curl } \mathbf{H}_g - \frac{\dot{N}}{N} \mathbf{E}_g$. Then we get

$$\text{curl} \left(\frac{c^2}{N} \text{curl } \mathbf{H}_g - \frac{\dot{N}}{N} \mathbf{E}_g \right) = -\left(\frac{1}{N}\right)^{\bullet\bullet} \mathbf{H}_g - 2\left(\frac{1}{N}\right)^{\bullet} \frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N} \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \quad (6.8)$$

The left-hand side here is

$$\begin{aligned} & \text{curl} \left(\frac{c^2}{N} \text{curl} \mathbf{H}_g - \frac{\dot{N}}{N} \mathbf{E}_g \right) \\ &= \text{grad} \frac{c^2}{N} \times \text{curl} \mathbf{H}_g + \frac{c^2}{N} \text{grad} \text{div} \mathbf{H}_g - \frac{c^2}{N} \Delta \mathbf{H}_g - \frac{\dot{N}}{N} \text{curl} \mathbf{E}_g - \text{grad} \frac{\dot{N}}{N} \times \mathbf{E}_g. \end{aligned}$$

From (6.4) we get $\text{div}(\mathbf{H}_g) = -N \mathbf{H}_g \text{grad} \left(\frac{1}{N} \right) \neq 0$. Thus, the nonlinear analogue of the wave equation is

$$\text{grad} \frac{c^2}{N} \times \text{curl} \mathbf{H}_g + \frac{c^2}{N} \text{grad} \text{div} \mathbf{H}_g - \frac{c^2}{N} \Delta \mathbf{H}_g - \frac{\dot{N}}{N} \text{curl} \mathbf{E}_g - \text{grad} \frac{\dot{N}}{N} \times \mathbf{E}_g \quad (6.9)$$

$$= - \left(\frac{1}{N} \right)^{\bullet\bullet} \mathbf{H}_g - 2 \left(\frac{1}{N} \right)^{\bullet} \frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N} \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \quad (6.10)$$

Note that if $N = \text{const}$, then we obtain the usual wave equation

$$\Delta \mathbf{H}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \quad (6.11)$$

Take now the constitutive equations in the form

$$\mathbf{D}_g = M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g, \quad (6.12)$$

$$\mathbf{H}_g = N \mathbf{B}_g - M \mathbf{E}_g, \quad (6.13)$$

where N, M are constants. In absence of sources, the Maxwell equations become

$$\text{curl} \mathbf{E}_g = - \frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div} \mathbf{B}_g = 0, \quad (6.14)$$

$$\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \quad \text{div} \mathbf{D}_g = 0. \quad (6.15)$$

If we express the Maxwell equations through \mathbf{E}_g and \mathbf{B}_g , then second pair of equations become

$$\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \quad (6.16)$$

$$N \text{curl} \mathbf{B}_g - M \text{curl} \mathbf{E}_g = M \frac{\partial \mathbf{B}_g}{\partial t} + \frac{N}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (6.17)$$

Since $\text{curl} \mathbf{E}_g = - \frac{\partial \mathbf{B}_g}{\partial t}$, from the last equation we get

$$\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (6.18)$$

The second equation, $\text{div} \mathbf{D}_g = 0$, reduces to $M \text{div} \mathbf{B}_g + \frac{N}{c^2} \text{div} \mathbf{E}_g = 0$. Since $\text{div} \mathbf{B}_g = 0$, we get

$$\text{div} \mathbf{E}_g = 0. \quad (6.19)$$

Thus, using constitutive equations with constant M and N we have Maxwell equations in terms of \mathbf{B}_g and \mathbf{E}_g , i.e.

$$\text{curl} \mathbf{E}_g = - \frac{\partial \mathbf{B}_g}{\partial t}, \quad \text{div} \mathbf{B}_g = 0, \quad (6.20)$$

$$\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}, \quad \text{div} \mathbf{E}_g = 0. \quad (6.21)$$

At this stage, we get the wave equations in the standard way. The time derivative of the first equation yields $\text{curl} \frac{\partial}{\partial t} \mathbf{E}_g = - \frac{\partial^2 \mathbf{B}_g}{\partial t^2} \Rightarrow c^2 \text{curl}(\text{curl} \mathbf{B}_g) = - \frac{\partial^2 \mathbf{B}_g}{\partial t^2}$. Since $\text{curl}(\text{curl} \mathbf{B}_g) = \text{grad}(\text{div} \mathbf{B}_g) - \Delta \mathbf{B}_g = -\Delta \mathbf{B}_g$, then

$$\Delta \mathbf{B}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}_g}{\partial t^2}. \quad (6.22)$$

By analogy $\text{curl} \frac{\partial}{\partial t} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2} \Rightarrow -\text{curl}(\text{curl} \mathbf{E}_g) = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2}$, and we get the wave equation for \mathbf{E}_g ,

$$\Delta \mathbf{E}_g = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_g}{\partial t^2}. \quad (6.23)$$

Thus, the gravi-electromagnetic waves \mathbf{E}_g and \mathbf{B}_g have speed c and do not depend on the constants M and N .

VII. WAVES AND CONSTITUTIVE EQUATIONS FOR LINEAR CONSTITUTIVE FUNCTIONS

Let us consider the constitutive equations (4.13)–(4.14) as linear functions of invariants, i.e.

$$M = M_g(I_{g1}, I_{g2}) = a_m I_{g1} + b_m I_{g2}, \quad (7.1)$$

$$N = N_g(I_{g1}, I_{g2}) = c^2 \varepsilon_g + a_n I_{g1} + b_n I_{g2}, \quad (7.2)$$

a_m, b_m, a_n, b_n being some constants. From all the Maxwell equations in material media, and in the absence of sources one finds $\text{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}$, $\text{curl}(N \mathbf{B}_g - M \mathbf{E}_g) = \frac{\partial}{\partial t} \left(M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g \right)$, and $N \text{curl} \mathbf{B}_g - M \text{curl} \mathbf{E}_g = M \frac{\partial \mathbf{B}_g}{\partial t} + \frac{N}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}$. Since $\text{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}$, from the last equation one gets

$$\text{curl} \mathbf{B}_g = \frac{1}{c^2} \frac{\partial \mathbf{E}_g}{\partial t}. \quad (7.3)$$

The second equation, $\text{div} \mathbf{D}_g = 0$, reduces to $\text{div} \left(M \mathbf{B}_g + \frac{N}{c^2} \mathbf{E}_g \right) = 0$, or $M \text{div} \mathbf{B}_g + \frac{N}{c^2} \text{div} \mathbf{E}_g = 0$. Since $\text{div} \mathbf{B}_g = 0$, one gets

$$\text{div} \mathbf{E}_g = 0. \quad (7.4)$$

VIII. INVERSE PROBLEM OF NONLINEAR GRAVITO-ELECTROMAGNETISM

In electrodynamics the direct solution of the Maxwell equations together with the nonlinear constitutive equations is a nontrivial and complicated task even for simple systems [7, 8]. In previous sections we presented some very special cases of the nonlinear functions N and M . Here we formulate the following inverse problem: if we have some particular solution of the gravity-Maxwell equations (4.8)–(4.9), can we then find the exact form of the corresponding nonlinear gravity-constitutive equations (4.13)–(4.14)?

It is natural to consider the case of plane gravitational waves, when the fields have only one space coordinate. We will show that even in this case one can have a nontrivial nonlinearity. Let us choose \mathbf{E}_g and \mathbf{B}_g mutually orthogonal and perpendicular to the direction of motion

$$\mathbf{E}_g = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_g = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}, \quad (8.1)$$

where $E \equiv E(t, y)$, $B \equiv B(t, y)$. Now the invariants (4.15) become

$$I_{g1} = B^2 - \frac{1}{c^2} E^2 \equiv I, \quad (8.2)$$

$$I_{g2} = 0. \quad (8.3)$$

The use of the nonlinear gravity-constitutive equations (4.13)–(4.14) gives for other fields

$$\mathbf{D}_g = \begin{pmatrix} \frac{1}{c^2} N E \\ 0 \\ M B \end{pmatrix}, \quad \mathbf{H}_g = \begin{pmatrix} -M E \\ 0 \\ N B \end{pmatrix}, \quad (8.4)$$

where $N \equiv N(I)$, $M \equiv M(I)$ are the sought for gravity-constitutive functions. They depend on I only, because of Lorentz invariance (see [7, 8]). Inserting the fields (8.1) and (8.4) into the gravity-Maxwell equations (4.8)–(4.9) without sources gives us 3 equations (hereafter, a prime with the corresponding subscript denotes the first partial derivative with respect to the variable in the subscript, while dot denotes time derivative)

$$E'_y = \dot{B}, \quad (8.5)$$

$$(NB)'_y = \frac{1}{c^2} (NE)' , \quad (8.6)$$

$$(ME)'_y = (MB)' . \quad (8.7)$$

Now we take into account that the gravity-constitutive functions N , M depend only on the invariant I and present (8.6)–(8.7) as the differential equations for them

$$N'_I \left(BI'_y - \frac{1}{c^2} E \dot{I} \right) + N \left(B'_y - \frac{1}{c^2} \dot{E} \right) = 0, \quad (8.8)$$

$$M'_I \left(EI'_y - B \dot{I} \right) = 0, \quad (8.9)$$

where we have exploited the identities

$$N'_y = N'_I I'_y, \quad M'_y = M'_I I'_y, \quad (8.10)$$

$$\dot{N} = N'_I \dot{I}, \quad \dot{M} = M'_I \dot{I}. \quad (8.11)$$

The second equation (8.9) can be immediately solved by

$$M(I) = \begin{cases} M_0 = \text{const}, & \text{if } EI'_y \neq B \dot{I}, \\ \text{arbitrary}, & \text{if } EI'_y = B \dot{I}. \end{cases} \quad (8.12)$$

The first equation (8.8) can be solved if

$$\lambda \equiv \frac{\left(B'_y - \frac{\dot{E}}{c^2} \right)}{\left(BI'_y - \frac{E \dot{I}}{c^2} \right)} \quad (8.13)$$

depends only on I , which is a very special case. One then has the differential equation

$$N'_I + \lambda(I) N = 0, \quad (8.14)$$

and its solution is

$$N(I) = N_0 e^{-\int \lambda(I) dI}. \quad (8.15)$$

Otherwise, by using the expressions for I'_y and \dot{I} from (8.2), i.e.

$$I'_y = 2BB'_y - \frac{2EE'_y}{c^2}, \quad \dot{I} = 2B\dot{B} - \frac{2E\dot{E}}{c^2}, \quad (8.16)$$

we obtain

$$2N'_I \left(B^2 B'_y + \frac{1}{c^4} E^2 \dot{E} - \frac{2}{c^2} E B E'_y \right) + N \left(B'_y - \frac{1}{c^2} \dot{E} \right) = 0, \quad (8.17)$$

where the sum of terms in brackets is not a function of I in general.

Usually, in the wave solutions the dependence of fields on frequency ω and wave number k is the same, and therefore we can consider the concrete choice

$$E(t, y) = f(\varepsilon \omega t + ky) \equiv f(X(t, y)), \quad B(t, y) = g(\varepsilon \omega t + ky) \equiv g(X(t, y)), \quad (8.18)$$

where $\varepsilon \equiv \pm 1$, with f and g arbitrary smooth nonvanishing functions. Bearing in mind that

$$E'_y = f'_X X'_y = k f'_X, \quad B'_y = g'_X X'_y = k g'_X,$$

$$\dot{E} = f'_X \dot{X} = \varepsilon \omega f'_X, \quad \dot{B} = g'_X \dot{X} = \varepsilon \omega g'_X,$$

our Eq. (8.5) yields

$$k f'_X = \varepsilon \omega g'_X. \quad (8.19)$$

Therefore

$$g(X) = \frac{k}{\varepsilon \omega} f(X) + \alpha, \quad (8.20)$$

where α is a constant, so that both E and B can be expressed through one function only, i.e. f , and the invariant I reads eventually as

$$I = \frac{1}{\omega^2} \left(k^2 - \frac{\omega^2}{c^2} \right) f^2 + 2 \frac{k}{\varepsilon \omega} \alpha f + \alpha^2. \quad (8.21)$$

The equations for the gravity-constitutive functions take therefore the form

$$N'_I \left[2I \left(k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{\omega^2}{c^2} \alpha^2 \right] + N \left(k^2 - \frac{\omega^2}{c^2} \right) = 0, \quad (8.22)$$

$$M'_I f'_X \left[\frac{2f}{\varepsilon \omega} \left(k^2 - \frac{\omega^2}{c^2} \right) + 2k\alpha \right] \alpha = 0, \quad (8.23)$$

where we have exploited the identities

$$g I'_y - \frac{f \dot{I}}{c^2} = \left(g f'_y - \frac{f \dot{f}}{c^2} \right) \left[\frac{2f}{\omega^2} \left(k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \alpha \right] \quad (8.24)$$

$$g f'_y - \frac{f \dot{f}}{c^2} = \frac{f'_X}{\varepsilon \omega} \left[f \left(k^2 - \frac{\omega^2}{c^2} \right) + k \varepsilon \omega \alpha \right], \quad (8.25)$$

and, after some cancellations,

$$\begin{aligned} & \left[f \left(k^2 - \frac{\omega^2}{c^2} \right) + k \varepsilon \omega \alpha \right] \left[\frac{2f}{\omega^2} \left(k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \alpha \right] \\ &= 2 \left(k^2 - \frac{\omega^2}{c^2} \right) I + 2 \frac{\omega^2}{c^2} \alpha^2, \end{aligned} \quad (8.26)$$

while

$$f I'_y - g \dot{I} = (f f'_y - g \dot{f}) \left[\frac{2f}{\omega^2} \left(k^2 - \frac{\omega^2}{c^2} \right) + 2 \frac{k}{\varepsilon \omega} \alpha \right], \quad (8.27)$$

$$f f'_y - g \dot{f} = f'_X (k f - \varepsilon \omega g) = -\varepsilon \omega f'_X \alpha. \quad (8.28)$$

The results of our analysis now depend on whether or not α vanishes. Indeed, if $\alpha = 0$, M is arbitrary and hence we obtain the equation

$$\left(k^2 - \frac{\omega^2}{c^2} \right) (2I N'_I + N) = 0, \quad (8.29)$$

which implies that either the dispersion relation

$$k^2 - \frac{\omega^2}{c^2} = 0 \quad (8.30)$$

holds, with N kept arbitrary, or such a dispersion relation is not fulfilled, while N is found from the differential equation

$$2IN_I' + N = 0, \quad (8.31)$$

which is solved by

$$N(I) = \frac{N_0}{\sqrt{I}}. \quad (8.32)$$

By contrast, if α does not vanish, M equals a constant M_0 , while N solves the more complicated equation (8.22). At this stage, to be consistent with the dependence of N on I only, we have to require again that the dispersion relation (8.30) should hold, jointly with $N_I' = 0$, which implies the constancy of N : $N = N_0$.

IX. CONCLUDING REMARKS

We have proposed a way to take an extremely general approach to a nonlinear formulation of gravity as classical electrodynamics. The framework formally includes nonLagrangian as well as Lagrangian theories, and accommodates the description of nonlocal effects. This is accomplished through generalized constitutive equations (and constitutive tensors). We expect future research directions to include the detailed development of new examples within this framework.

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